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Derivation of General Formula for Integration of Powers of Hyperbolic Sine and Hyperbolic Cosine Functions through Binomial Theorem

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Abstract
Aim: The objective of this paper was to derive a general formula for the integration of powers of hyperbolic sine and hyperbolic cosine functions using the Binomial Theorem.
Methodology: The study used pure mathematical research to develop formulas by discovering through mathematical methods and theorems that lead to the generalization of the formulas presented.
Results: The derived formulas for integrating powers of hyperbolic sine and hyperbolic cosine functions using the Binomial Theorem are presented. Additionally, general formulas for integrating both odd and even powers of these hyperbolic functions are derived. The application of these generalized formulas for odd and even powers is illustrated to enhance understanding.
Conclusion: The derivation of the integration formulas for powers of hyperbolic sine and hyperbolic cosine functions has led to the development of generalized formulas for both odd and even powers. These generalized formulas simplify the process of integrating powers of hyperbolic sine and hyperbolic cosine functions, providing more efficient solutions.

Keywords: power of hyperbolic sine function, power of hyperbolic cosine function, hyperbolic identities, exponential function, Binomial theorem

INTRODUCTION

Integrating hyperbolic functions is an essential aspect of studying integral calculus. In many ways, hyperbolic functions are analogous to trigonometric functions but are defined using a hyperbola rather than a unit circle. There are six hyperbolic functions in terms of x: hyperbolic sine x (sinh x), hyperbolic cosine x (cosh x), hyperbolic tangent x (tanh x), hyperbolic cosecant x (csch x), hyperbolic secant x (sech x), and hyperbolic cotangent x (coth x). Hyperbolic functions are applied in studying catenary, light, electricity, velocity, etc. (Stewart., et.al, 2019; Thomas, 2004). The study of Baskonus and Bulut (2015) presented hyperbolic functions solutions for differential equations while Stakhov and Rozin (2004) presented a new class of hyperbolic functions that unite the characteristics of the classical hyperbolic functions and the recurring Fibonacci and Lucas series.

Researchers in mathematics often highlight the applications of hyperbolic functions in various fields such as science and engineering. However, many students find it challenging to relate to these applications due to the complexity of integrating hyperbolic functions. According to Yerizon (2019), students face several difficulties when learning calculus, including errors in integrating negative-level functions, fractional functions, and the process of reducing the assumed function. Students also struggle with changing the form of the results into the required form by the original question and making substitution errors.

Domondon et.al (2022), stated that students find Basic Calculus difficult, especially along Integrals. Moreover, some of the difficulties the students experienced were the lack of knowledge of the concepts, poor application, complicated formulas and processes, and confusion in understanding the problem. Whereas, Kiat (2005) observed that students tend to focus more on the procedural aspects of integration rather than the conceptual ones. Both conceptual and procedural understanding of integration is necessary, but many students commit technical errors due to a lack of specific mathematical content knowledge.

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Numerous techniques have been developed to solve integration problems in calculus. Some methods used to assess integrals of hyperbolic functions include substitution and change of variables, integration by parts, trigonometric integrals, hyperbolic substitutions, and miscellaneous substitution.

Alcantara (2015) used tabular technique of integration by parts to evaluate the integrals of products of elementary functions involving hyperbolic functions. This technique made it simple and easy to take the integral of product of integrand which has hyperbolic functions.

Whereas Dar and Paris (2019), used a hypergeometric approach by evaluating some integrals over [0,∞) of quotients of powers of the hyperbolic functions sinh x and cosh x. Some of these results appear to be new but several verify the entries in the table of integrals of Gradshteyn and Rhyzik.

While many integration formulas can be derived from their corresponding derivative formulas, some integration problems require more complex approaches. For example, to evaluate the integral of $\int sinh^{13}\, 3x\, dx$, one needs to do the following procedures;

$$\int sinh^{13}\, 3x\, dx = \int sinh^{12}\, 3x\, (sinh\, 3x\ \ dx)$$

$$\int sinh^{13}\, 3x\, dx = \int (sinh^{2}\, 3x)^{6}\, (sinh\, 3x\ \ dx)$$

use the identities,

$$cosh^{2}\, x - sinh^{2}\, x = 1$$

$$sinh^{2}\, x = cosh^{2}\, x - 1$$

$$\int sinh^{13}\, 3x\, dx = \int (cosh^{2}\, 3x - 1)^{6}\, (sinh\, 3x\ \ dx)$$

then expand the power of 6,

$$\int sinh^{13}\, 3x\, dx = \int (cosh^{12}\, 3x - 6cosh^{10}\, 3x + 15cosh^{8}\, 3x - 20cosh^{6}\, 3x + 15cosh^{4}\, 3x - 6cosh^{2}\, 3x + 1)\ (sinh\, 3x\ \ dx)$$

distribute $\sinh 3x\ \ dx$,

$$\int sinh^{13}\, 3x\, dx = \int cosh^{12}\ 3x\, sinh\, 3x\ \ dx - 6\int cosh^{10}\, 3x\, sinh\, 3x\, dx + 15\int cosh^{8}\ 3x\, sinh\, 3x\, dx - 20\int cosh^{6}\, 3x\, sinh\, 3x\, dx + 15\int cosh^{4}\ 3x\, sinh\, 3x\, dx - 6\int cosh^{2}\, 3x\, sinh\, 3x\, dx + \int sinh\, 3x\, dx$$

by substitution, let u = cosh 3x, du = (sinh 3x)/3 and then,

$$\int sinh^{13}\, 3x\, dx = \frac{1}{39}\, cosh^{13}\, 3x - \frac{6}{33} cosh^{11}\, 3x + \frac{15}{27}\, cosh^{9}\, 3x - \frac{20}{21}\, cosh^{7}\, 3x + cosh^{5}\, 3x - \frac{6}{9}\, cosh^{3}\, 3x + \frac{1}{3} \cosh 3x + C$$

As shown in the solution above, the power of sine is reduced, hyperbolic identities and substitutions are used then the power is expanded. Several students will get tired of solving some of the integrals that have long and tedious solutions like taking the integrals of powers of hyperbolic sine and hyperbolic cosine functions. This long and tedious solutions can be solved in a much simpler way and faster even the power of hyperbolic functions are too high by deriving a formula through expressing the hyperbolic functions in terms of the exponential function, e^{x}. The hyperbolic sine x can be defined as $\sinh x = \frac{e^{x} - e^{-x}}{2}$ while hyperbolic cosine x as $\cosh x = \frac{e^{x} + e^{-x}}{2}$ (Stewart., et.al, 2019; Thomas, 2004). This paper expanded the use of Binomial Theorem and basic integration method to integrals of powers of hyperbolic functions.

The algebraic expression of the form a + b is called binomial. Binomial with small positive power can be solved by basic multiplication but raising it to a very high positive power would be tedious. By applying Binomial Theorem this

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can be solved in a much simpler and shorter way. Binomial theorem is defined as the rule or formula for expansion of $(a + b)^n$, where n is any positive integral power, For any positive integer n

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \binom{n}{3}a^{n-3}b^3 + \cdots + \binom{n}{r}a^{n-r}b^r \quad \textbf{or}$$

$(a+b)^n = \sum_{r=0}^{n}\binom{n}{r}a^{n-r}b^r$ (Stewart et.l, 2012). The researcher employed this theorem to have simpler and easier way in expanding and integrating the power of hyperbolic sine and hyperbolic cosine functions.

This study is only limited to evaluating the powers of hyperbolic sine and hyperbolic cosine integrals and assessing formulas that include the power of hyperbolic sine and hyperbolic cosine functions.

Objectives

The main objective of the paper was to derive a formula for the integration of the power of hyperbolic sine and hyperbolic cosine functions by the Binomial Theorem.

Specifically, it also aimed to:

1. derive a general formula for odd and even power of hyperbolic sine and hyperbolic cosine functions,
2. illustrate the application of the derived formulas of odd and even power of hyperbolic sine and hyperbolic cosine functions.

The results of the study were useful to the students who are taking Calculus for they can use the derived formula in evaluating the integrals that involve the power of hyperbolic sine and hyperbolic cosine functions.

METHODS

Research Design

This study employed a descriptive research design, focusing on the development of formulas for integrating powers of hyperbolic sine and cosine functions through mathematical methods and theorems.

Population and Sampling

The study did not involve a specific population or sampling procedure, as it was based on mathematical analysis rather than empirical data collection.

Instrument

The primary instrument used in this study was mathematical analysis, involving the application of the Binomial Theorem and basic integration techniques. The Binomial Theorem is used to expand the power of hyperbolic sine and hyperbolic cosine functions. The hyperbolic identities are used after the expansion and regrouping of the terms. And the basic integration of hyperbolic sine x and hyperbolic cosine x is employed.

Data Collection

Data collection in this study involved the identification and derivation of mathematical formulas for integrating hyperbolic functions. Theoretical derivations were supported by rigorous mathematical proofs.

Treatment of Data

The data, in this case, mathematical formulas, were analyzed and interpreted to derive the general formula for integrating powers of hyperbolic sine and cosine functions.

Ethics in Research

Since this study did not involve human subjects or data, ethical considerations related to human subjects were not applicable.

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RESULTS and DISCUSSION

Derivation of General Formula for the Integration of Powers of Hyperbolic Sine and Hyperbolic Cosine Functions

To develop integration formulas for powers of hyperbolic sine and hyperbolic cosine functions, the following formulas were derived.

Integration of the Power of Hyperbolic Sine

Given $\int sinh^n\, ax\, dx$, where n is any positive integer and a is any real number except 0. Since $sinh\, x = \frac{e^x - e^{-x}}{2}$, then

$$\int sinh^n\, ax\, dx = \int \left(\frac{e^{ax} - e^{-ax}}{2}\right)^n dx$$

take out $\left(\frac{1}{2}\right)^n$, it gives

$$\int sinh^n\, ax\, dx == \left(\frac{1}{2}\right)^n \int (e^{ax} - e^{-ax})^n\, dx$$

Then expand n in the integrand by Binomial theorem

$$\int sinh^n\, ax\, dx = \left(\frac{1}{2}\right)^n \int \Bigg\{ e^{axn} - \binom{n}{1} e^{ax(n-1)} e^{-ax} + \binom{n}{2} e^{ax(n-2)} e^{-2ax} - \binom{n}{3} e^{ax(n-3)} e^{-3ax}$$
$$+ \binom{n}{4} e^{ax(n-4)} e^{-4ax} - \ \ldots + \binom{n}{\frac{n}{2}} e^{ax\left(\frac{n}{2}\right)} e^{-ax\left(\frac{n}{2}\right)} - \cdots + \binom{n}{\frac{n-1}{2}} e^{ax} - \binom{n}{\frac{n+1}{2}} e^{-a} \ + \cdots$$
$$\pm \binom{n}{n-3} e^{3ax} e^{-ax(n-3)} \pm \binom{n}{n-2} e^{2ax} e^{-ax(n-2)} \pm \binom{n}{n-1} e^{ax} e^{-ax(n-1)} \pm e^{-axn} \Bigg\} dx$$

simplify,

$$\int sinh^n\, ax\, dx = \left(\frac{1}{2}\right)^n \int \Bigg\{ e^{axn} - \binom{n}{1} e^{ax(n-2)} + \binom{n}{2} e^{ax(n-4)} - \binom{n}{3} e^{ax(n-6)} + \binom{n}{4} e^{ax(n-8)} - \cdots + \binom{n}{\frac{n}{2}} e^{ax(0)}$$
$$- \cdots + \binom{n}{\frac{n-1}{2}} e^{ax} - \binom{n}{\frac{n+1}{2}} e^{-ax} + \cdots \pm \binom{n}{n-3} e^{-ax(n-6)} \pm \binom{n}{n-2} e^{-ax(n-4)}$$
$$\pm \binom{n}{n-1} e^{-ax(n-2)} \pm e^{-axn} \Bigg\} dx \qquad eq.(\mathbf{1})$$

Odd Powers of Hyperbolic Sine

As shown in equation (1), the coefficient $\binom{n}{\frac{n-1}{2}}$ indicates odd power. With this, the general formula for integration of odd power of hyperbolic sine can be derived.

$$\int sinh^n\, ax\, dx = \left(\frac{1}{2}\right)^n \int \Bigg\{ e^{axn} - \binom{n}{1} e^{ax(n-2)} + \binom{n}{2} e^{ax(n-4)} - \binom{n}{3} e^{ax(n-6)} + \binom{n}{4} e^{ax(n-8)} - \cdots$$
$$+ \binom{n}{\frac{n-1}{2}} e^{ax} - \binom{n}{\frac{n+1}{2}} e^{-ax} + \cdots + \binom{n}{n-3} e^{-ax(n-6)} - \binom{n}{n-2} e^{-ax(n-4)}$$
$$+ \binom{n}{n-1} e^{-a\ (n-2)} - e^{-ax} \Bigg\} dx$$

Since $\binom{n}{1} = \binom{n}{n-1}$, $\binom{n}{2} = \binom{n}{n-2}$, …. , regroup the terms in the integrand

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$$\int sinh^n\, ax\, dx = \left(\frac{1}{2}\right)^n \int \Bigg\{ (e^{axn} - e^{-axn}) - \binom{n}{1}\left[e^{ax(n-2)} - e^{-ax(n-2)}\right] + \binom{n}{2}\left[e^{ax(n-4)} - e^{-ax(n-4)}\right] - \binom{n}{3}\left[e^{ax(n-6)} - e^{-a\ (n-6)}\right] + \cdots + \binom{n}{\frac{n-1}{2}}\left[e^{ax} - e^{-ax}\right]\Bigg\} dx$$

Take out $\frac{1}{2}$ in $\left(\frac{1}{2}\right)^n$, and distribute in the integrand,

$$\int sinh^n\, ax\, dx = \left(\frac{1}{2}\right)^{n-1} \int \Bigg\{ \left(\frac{e^{axn} - e^{-axn}}{2}\right) - \binom{n}{1}\left[\frac{e^{ax(n-2)} - e^{-ax(n-2)}}{2}\right] + \binom{n}{2}\left[\frac{e^{ax(n-4)} - e^{-ax(n-4)}}{2}\right] - \binom{n}{3}\left[\frac{e^{ax(n-6)} - e^{-ax(n-6)}}{2}\right] + \cdots + \binom{n}{\frac{n-1}{2}}\left[\frac{e^{ax} - e^{-a}}{2}\right]\Bigg\} dx$$

Since $sinh\, ax = \frac{e^{ax} - e^{-ax}}{2}$, then the equation above can be written as

$$\int sinh^n\, ax\, dx = \left(\frac{1}{2}\right)^{n-1} \int \Bigg\{ \sinh axn - \binom{n}{1}[\sinh ax(n-2)] + \binom{n}{2}[\sinh ax(n-4)] - \binom{n}{3}[\sinh ax(n-6)] + - \cdots + \binom{n}{\frac{n-1}{2}}[\sinh ax(n-6)]\Bigg\} dx$$

Then integrate, it gives the derived formula for the odd power of hyperbolic sine:

$$\int sinh^n\, ax\, dx = \left(\frac{1}{2}\right)^{n-1} \Bigg\{ \frac{1}{an}\cosh axn - \binom{n}{1}\frac{1}{a(n-2)}[\cosh ax(n-2)] + \binom{n}{2}\frac{1}{a(n-4)}[\cosh ax(n-4)] - \binom{n}{3}\frac{1}{a(n-6)}[\cosh ax(n-6)] + \cdots + \binom{n}{\frac{n-1}{2}}\frac{1}{a}[\cosh ax]\Bigg\} + C \qquad eq. (\mathbf{2})$$

From equation (2), factor out $\frac{1}{a}$, then the general integration formula for the integral of the odd powers of hyperbolic sine is

$$\int sinh^n\, ax\, dx = \left(\frac{1}{a}\right)\left(\frac{1}{2}\right)^{n-1} \Bigg\{ \frac{1}{n}\cosh axn - \binom{n}{1}\left[\frac{1}{(n-2)}\cosh ax(n-2)\right] + \binom{n}{2}\left[\frac{1}{(n-4)}\cosh ax(n-4)\right] - \binom{n}{3}\left[\frac{1}{(n-6)}\cosh ax(n-6)\right] + \cdots + \binom{n}{\frac{n-1}{2}}[\cosh ax]\Bigg\} + C$$

where n is an odd positive integer, $n \geq 3$ and a is any real number except 0. In symbols:

$$\int sinh^n\, ax\, dx = \left(\frac{1}{a}\right)\left(\frac{1}{2}\right)^{n-1}\left[\sum_{k}^{\frac{n-1}{2}} (-1)^k \binom{n}{k}\frac{1}{(n-2k)} cosh\, a(n-2k)x\right] + C \qquad eq. (\mathbf{3})$$

where $k = 0, 1, 2, 3, \ldots \frac{n-1}{2}$, $\quad n \geq 3$, $\quad \binom{n}{k} = \frac{n!}{k!(n-k)}$

Even Powers of Hyperbolic Sine

As shown in equation (1), the coefficient $\binom{n}{\frac{n}{2}}$ indicates when the exponent n is an even number. With this, the general formula for integration of even power of hyperbolic sine can be derived.

$$\int sinh^n\, ax\, dx = \left(\frac{1}{2}\right)^n \int \Bigg\{ e^{axn} - \binom{n}{1}e^{ax(n-2)} + \binom{n}{2}e^{ax(n-4)} - \binom{n}{3}e^{ax(n-6)} + \binom{n}{4}e^{ax(n-8)} + \cdots - \binom{n}{\frac{n}{2}}e^{ax(0)} + \cdots - \binom{n}{n-3}e^{-a\ (n-6)} + \binom{n}{n-2}e^{-ax(n-4)} - \binom{n}{n-1}e^{-a\ (n-2)} + e^{-axn}\Bigg\} dx$$

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Since $\binom{n}{1} = \binom{n}{n-1}$, $\binom{n}{2} = \binom{n}{n-2}$, ... , regroup the terms in the integrand

$$\int sinh^n\, ax\, dx = \left(\frac{1}{2}\right)^n \int \Big\{ (e^{axn} + e^{-axn}) - \binom{n}{1}\left[e^{ax(n-2)} + e^{-ax(n-2)}\right] + \binom{n}{2}\left[e^{ax(n-4)} + e^{-ax(n-4)}\right]$$
$$- \binom{n}{3}\left[e^{ax(n-6)} + e^{-a\ (n-6)}\right] + \cdots + \binom{n}{\frac{n}{2}} \Big\} dx$$

Then take out $\frac{1}{2}$ in $\left(\frac{1}{2}\right)^n$ and distribute in the integrand,

$$\int sinh^n\, ax\, dx = \left(\frac{1}{2}\right)^{n-1} \int \left\{ \left(\frac{e^{axn} + e^{-axn}}{2}\right) - \binom{n}{1}\left[\frac{e^{ax(n-2)} + e^{-ax(n-2)}}{2}\right] + \binom{n}{2}\left[\frac{e^{ax(n-4)} + e^{-ax(n-4)}}{2}\right] \right.$$
$$\left. - \binom{n}{3}\left[\frac{e^{ax(n-6)} + e^{-ax(n-6)}}{2}\right] + \cdots + \left(\frac{1}{2}\right)\binom{n}{\frac{n}{2}} \right\} dx$$

Since $cosh\, ax = \frac{e^{ax} + e^{-ax}}{2}$, then the equation above can be written as

$$\int sinh^n\, ax\, dx = \left(\frac{1}{2}\right)^{n-1} \int \Big\{ \cosh axn - \binom{n}{1}[\cosh ax(n-2)] + \binom{n}{2}[\cosh ax(n-4)] - \binom{n}{3}[\cosh ax(n-6)]$$
$$+ - \cdots + \left(\frac{1}{2}\right)\binom{n}{\frac{n}{2}} \Big\} dx$$

Then integrate, it gives the derived formula for the even power of hyperbolic sine:

$$\int sinh^n\, ax\, dx = \left(\frac{1}{2}\right)^{n-1} \Big\{ \frac{1}{an} \sinh axn - \binom{n}{1}\frac{1}{a(n-2)}[\sinh ax(n-2)] + \binom{n}{2}\frac{1}{a(n-4)}[\sinh ax(n-4)]$$
$$- \binom{n}{3}\frac{1}{a(n-6)}[\sinh ax(n-6)] + \cdots + \left(\frac{1}{2}\right)\binom{n}{\frac{n}{2}} x \Big\} + C \qquad eq.(\mathbf{4})$$

From equation (4), factor out $\frac{1}{a}$ in the sine series, the results gives the generalized formula for the integral of the even powers of hyperbolic sine

$$\int sinh^n\, ax\, dx = \left(\frac{1}{2}\right)^{n-1} \frac{1}{a} \Big\{ \frac{1}{n} \sinh axn - \binom{n}{1}\frac{1}{(n-2)}[\sinh ax(n-2)] + \binom{n}{2}\frac{1}{(n-4)}[\sinh ax(n-4)]$$
$$- \binom{n}{3}\frac{1}{(n-6)}[\sinh ax(n-6)] + \cdots \Big\} + \left(\frac{1}{2}\right)\binom{n}{\frac{n}{2}} x + C$$

where n is an even positive integer, $n \geq 2$ and a is any real number except 0. In symbols:

$$\int sinh^n\, ax\, dx = (-1)^k \left\{ \left(\frac{1}{a}\right)\left(\frac{1}{2}\right)^{n-1} \left[\sum_{k}^{\frac{n-2}{2}} \binom{n}{k} \frac{1}{(n-2k)} sinh\, a(n-2k)x \right] + \left[\left(\frac{1}{2}\right)\binom{n}{k} x \right] \right\} + C \qquad eq.(\mathbf{5})$$

where $k = 0, 1, 2, 3, \ldots \frac{n-2}{2}$, $\quad n \geq 2$, $\quad \binom{n}{k} = \frac{n!}{k!(n-k)}$,

Integration of the Power of Hyperbolic Cosine

Given $\int cosh^n\, ax\, dx$, where n is any positive integer and a is any real number except 0. Since $cosh\, x = \frac{e^{x} + e^{-x}}{2}$, then

$$\int cosh^n\, ax\, dx = \int \left(\frac{e^{ax} + e^{-ax}}{2}\right)^n dx$$

take out $\left(\frac{1}{2}\right)^n$, it gives

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$$\int cosh^n\ ax\ dx == \left(\frac{1}{2}\right)^n \int (e^{ax} + e^{-a}\ )^n\ dx$$

Then expand n in the integrand by Binomial theorem

$$\int cosh^n\ ax\ dx = \left(\frac{1}{2}\right)^n \int \Bigg\{ e^{axn} + \binom{n}{1} e^{ax(n-1)} e^{-ax} + \binom{n}{2} e^{ax(n-2)} e^{-2ax} + \binom{n}{3} e^{ax(n-3)} e^{-3ax}$$
$$+ \binom{n}{4} e^{ax(n-4)} e^{-4a} + \ldots + \binom{n}{\frac{n}{2}} e^{ax\left(\frac{n}{2}\right)} e^{-ax\left(\frac{n}{2}\right)} + \cdots + \binom{n}{\frac{n-1}{2}} e^{ax} + \binom{n}{\frac{n+1}{2}} e^{-ax} + \cdots$$
$$+ \binom{n}{n-3} e^{3a}\ e^{-a\ (n-3)} + \binom{n}{n-2} e^{2a}\ e^{-ax(n-2)} + \binom{n}{n-1} e^{ax} e^{-ax(n-1)} + e^{-axn} \Bigg\} dx$$

simplify,

$$\int cosh^n\ ax\ dx = \left(\frac{1}{2}\right)^n \int \Bigg\{ e^{axn} + \binom{n}{1} e^{ax(n-2)} + \binom{n}{2} e^{ax(n-4)} + \binom{n}{3} e^{ax(n-6)} + \binom{n}{4} e^{ax(n-8)} + \cdots + \binom{n}{\frac{n}{2}} e^{ax(0)}$$
$$+ \cdots + \binom{n}{\frac{n-1}{2}} e^{ax} + \binom{n}{\frac{n+1}{2}} e^{-ax} + \cdots + \binom{n}{n-3} e^{-a\ (n-6)} + \binom{n}{n-2} e^{-ax(n-4)}$$
$$+ \binom{n}{n-1} e^{-ax(n-2)} + e^{-axn} \Bigg\} dx \qquad eq. (\mathbf{6})$$

Odd Powers of Hyperbolic Cosine

As shown in equation (6), the coefficient $\binom{n}{\frac{n-1}{2}}$ indicates odd power. With this, the general formula for integration of odd power of hyperbolic cosine can be derived.

$$\int cosh^n\ ax\ dx = \left(\frac{1}{2}\right)^n \int \Bigg\{ e^{axn} + \binom{n}{1} e^{ax(n-2)} + \binom{n}{2} e^{ax(n-4)} + \binom{n}{3} e^{ax(n-6)} + \binom{n}{4} e^{ax(n-8)} + \cdots$$
$$+ \binom{n}{\frac{n-1}{2}} e^{ax} + \binom{n}{\frac{n+1}{2}} e^{-a}\ + \cdots + \binom{n}{n-3} e^{-ax(n-6)} + \binom{n}{n-2} e^{-ax(n-4)}$$
$$+ \binom{n}{n-1} e^{-ax(n-2)} + e^{-axn} \Bigg\} dx$$

Since $\binom{n}{1} = \binom{n}{n-1}$, $\binom{n}{2} = \binom{n}{n-2}$, ... , regroup the terms in the integrand

$$\int cosh^n\ ax\ dx = \left(\frac{1}{2}\right)^n \int \Bigg\{ (e^{axn} + e^{-axn}) - \binom{n}{1} \left[e^{ax(n-2)} + e^{-ax(n-2)}\right] + \binom{n}{2} \left[e^{ax(n-4)} + e^{-ax(n-4)}\right]$$
$$+ \binom{n}{3} \left[e^{ax(n-6)} + e^{-ax(n-6)}\right] + \cdots + \binom{n}{\frac{n-1}{2}} \left[e^{ax} + e^{-ax}\right] \Bigg\} dx$$

Take out $\frac{1}{2}$ in $\left(\frac{1}{2}\right)^n$, and distribute in the integrand,

$$\int cosh^n\ ax\ dx = \left(\frac{1}{2}\right)^{n-1} \int \Bigg\{ \left(\frac{e^{axn} + e^{-ax}}{2}\right) + \binom{n}{1} \left[\frac{e^{ax(n-2)} + e^{-ax(n-2)}}{2}\right] + \binom{n}{2} \left[\frac{e^{ax(n-4)} + e^{-ax(n-4)}}{2}\right]$$
$$+ \binom{n}{3} \left[\frac{e^{ax(n-6)} + e^{-ax(n-6)}}{2}\right] + \cdots + \binom{n}{\frac{n-1}{2}} \left[\frac{e^{ax} + e^{-ax}}{2}\right] \Bigg\} dx$$

Since $cosh\ ax = \frac{e^{ax} + e^{-ax}}{2}$, then the equation above can be written as

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$$\int cosh^n\, ax\, dx = \left(\frac{1}{2}\right)^{n-1} \int \Bigg\{ \cosh axn + \binom{n}{1}[\cosh ax(n-2)] + \binom{n}{2}[\cosh ax(n-4)] + \binom{n}{3}[\cosh ax(n-6)] + \cdots + \binom{n}{\frac{n-1}{2}}[\cosh ax(n-6)] \Bigg\} dx$$

Then integrate, it gives the derived formula for the odd power of hyperbolic sine:

$$\int cosh^n\, ax\, dx = \left(\frac{1}{2}\right)^{n-1} \Bigg\{ \frac{1}{an}\sinh axn + \binom{n}{1}\frac{1}{a(n-2)}[\sinh ax(n-2)] + \binom{n}{2}\frac{1}{a(n-4)}[\sinh ax(n-4)] - \binom{n}{3}\frac{1}{a(n-6)}[\sinh ax(n-6)] + \cdots + \binom{n}{\frac{n-1}{2}}\frac{1}{a}[\sinh ax] \Bigg\} + C \qquad eq.(\mathbf{7})$$

From equation (7), factor out $\frac{1}{a}$, then the general integration formula for the integral of the odd powers of hyperbolic cosine is

$$\int cosh^n\, ax\, dx = \left(\frac{1}{a}\right)\left(\frac{1}{2}\right)^{n-1} \Bigg\{ \frac{1}{n}\sinh axn + \binom{n}{1}\frac{1}{(n-2)}[\sinh ax(n-2)] + \binom{n}{2}\frac{1}{(n-4)}[\sinh ax(n-4)] - \binom{n}{3}\frac{1}{(n-6)}[\sinh ax(n-6)] + \cdots + \binom{n}{\frac{n-1}{2}}[\sinh ax] \Bigg\} + C$$

where n is an odd positive integer, $n \geq 3$ and a is any real number except 0. In symbols:

$$\int cosh^n\, ax\, dx = \left(\frac{1}{a}\right)\left(\frac{1}{2}\right)^{n-1} \left[\sum_{k}^{\frac{n-1}{2}} \binom{n}{k} \frac{1}{(n-2k)} sinh\, a(n-2k)x \right] + C \qquad eq.(\mathbf{8})$$

where $k = 0, 1, 2, 3, \ldots \frac{n-1}{2}$, $\quad n \geq 3$, $\quad \binom{n}{k} = \frac{n!}{k!(n-k)}$

Even Powers of Hyperbolic Cosine

As shown in equation (6), the coefficient $\binom{n}{\frac{n}{2}}$ indicates when the exponent n is an even number. With this, the general formula for integration of even power of hyperbolic cosine can be derived.

$$\int cosh^n\, ax\, dx = \left(\frac{1}{2}\right)^{n} \int \Bigg\{ e^{axn} + \binom{n}{1}e^{ax(n-2)} + \binom{n}{2}e^{ax(n-4)} + \binom{n}{3}e^{ax(n-6)} + \binom{n}{4}e^{ax(n-8)} + \cdots + \binom{n}{\frac{n}{2}}e^{ax(0)} + \cdots + \binom{n}{n-3}e^{-ax(n-6)} + \binom{n}{n-2}e^{-ax(n-4)} + \binom{n}{n-1}e^{-ax(n-2)} + e^{-axn} \Bigg\} dx$$

Since $\binom{n}{1} = \binom{n}{n-1}$, $\binom{n}{2} = \binom{n}{n-2}$, …. , regroup the terms in the integrand

$$\int cosh^n\, ax\, dx = \left(\frac{1}{2}\right)^{n} \int \Bigg\{ (e^{axn} + e^{-axn}) + \binom{n}{1}\left[e^{ax(n-2)} + e^{-a\ (n-2)}\right] + \binom{n}{2}\left[e^{ax(n-4)} + e^{-ax(n-4)}\right] + \binom{n}{3}\left[e^{ax(n-6)} + e^{-ax(n-6)}\right] + \cdots + \binom{n}{\frac{n}{2}} \Bigg\} dx$$

Then take out $\frac{1}{2}$ in $\left(\frac{1}{2}\right)^n$ and distribute in the integrand,

$$\int cosh^n\, ax\, dx = \left(\frac{1}{2}\right)^{n-1} \int \Bigg\{ \left(\frac{e^{axn} + e^{-axn}}{2}\right) + \binom{n}{1}\left[\frac{e^{ax(n-2)} + e^{-ax(n-2)}}{2}\right] + \binom{n}{2}\left[\frac{e^{ax(n-4)} + e^{-ax(n-4)}}{2}\right] + \binom{n}{3}\left[\frac{e^{ax(n-6)} + e^{-ax(n-6)}}{2}\right] + \cdots + \left(\frac{1}{2}\right)\binom{n}{\frac{n}{2}} \Bigg\} dx$$

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Since $cosh\,ax = \frac{e^{ax}+e^{-ax}}{2}$, then the equation above can be written as

$$\int cosh^n\,ax\,dx = \left(\frac{1}{2}\right)^{n-1} \int \left\{ \cosh axn + \binom{n}{1}[\cosh ax(n-2)] + \binom{n}{2}[\cosh ax(n-4)] - \binom{n}{3}[\cosh ax(n-6)] + \cdots + \left(\frac{1}{2}\right)\binom{n}{\frac{n}{2}} \right\} dx$$

Then integrate, it gives the derived formula for the even power of hyperbolic sine:

$$\int cosh^n\,ax\,dx = \left(\frac{1}{2}\right)^{n-1} \left\{ \frac{1}{an}\sinh axn + \binom{n}{1}\frac{1}{a(n-2)}[\sinh ax(n-2)] + \binom{n}{2}\frac{1}{a(n-4)}[\sinh ax(n-4)] + \binom{n}{3}\frac{1}{a(n-6)}[\sinh ax(n-6)] + \cdots + \left(\frac{1}{2}\right)\binom{n}{\frac{n}{2}} x \right\} + C \qquad eq.(\mathbf{9})$$

From equation (9), factor out $\frac{1}{a}$ in the sine series, the results gives the generalized formula for the integral of the even powers of hyperbolic cosine

$$\int cosh^n\,ax\,dx = \left(\frac{1}{2}\right)^{n-1} \frac{1}{a}\left\{ \frac{1}{n}\sinh axn - \binom{n}{1}\frac{1}{(n-2)}[\sinh ax(n-2)] + \binom{n}{2}\frac{1}{(n-4)}[\sinh ax(n-4)] - \binom{n}{3}\frac{1}{(n-6)}[\sinh ax(n-6)] + \cdots \right\} + \left(\frac{1}{2}\right)\binom{n}{\frac{n}{2}} x + C$$

where n is an even positive integer, $n \geq 2$ and a is any real number except 0. In symbols:

$$\int cosh^n\,ax\,dx = (-1)^k \left\{ \left(\frac{1}{a}\right)\left(\frac{1}{2}\right)^{n-1} \left[\sum_{k}^{\frac{n-2}{2}} \binom{n}{k} \frac{1}{(n-2k)} sinh\,a(n-2k)x \right] + \left[\left(\frac{1}{2}\right)\binom{n}{k} x \right] \right\} + C \qquad eq.(\mathbf{10})$$

where $k = 0, 1, 2, 3, \ldots \frac{n-2}{2}$, $\quad n \geq 2, \quad \binom{n}{k} = \frac{n!}{k!(n-k)}$,

APPLICATIONS

The following illustrates the application of the derived generalized formula for integrating the powers of the hyperbolic sine and the hyperbolic cosine functions.

Example 1: Evaluate $\int \sinh^{13} 3x\,dx$ using long method and the derived formula for odd power of hyperbolic sine

By long method:

Substitute $sinh\,x = \frac{e^x - e^{-x}}{2}$ to the given, then

$$\int sinh^{13}\,3x\,dx = \int \left(\frac{e^{3x} - e^{-3x}}{2}\right)^{13} dx$$

take out $\left(\frac{1}{2}\right)^{13}$, it gives

$$\int sinh^{13}\,3x\,dx == \left(\frac{1}{2}\right)^{13} \int (e^{3x} - e^{-3x})^{13}\,dx$$

Then expand n=13 in the integrand by Binomial theorem

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$$\int sinh^{13}\, 3x\, dx = \left(\frac{1}{2}\right)^{13} \int \left\{ e^{3(13)x} - \binom{13}{1} e^{3(12)x}e^{-3x} + \binom{13}{2} e^{3(11)x}e^{-3(2)x} - \binom{13}{3} e^{3(10)x}e^{-3(3)x} + \binom{13}{4} e^{3(9)x}e^{-3(4)x} - \binom{13}{5} e^{3(8)x}e^{-3(5)x} + \binom{13}{6} e^{3(7)x}e^{-3(6)x} - \binom{13}{7} e^{3(6)x}e^{-3(7)x} + \binom{13}{8} e^{3(5)x}e^{-3(8)x} - \binom{13}{9} e^{3(4)x}e^{-3(9)x} + \binom{13}{10} e^{3(3)x}e^{-3(10)x} - \binom{13}{11} e^{3(2)x}e^{-3(11)x} + \binom{13}{12} e^{3x}e^{-3(12)x} - e^{-3(13)x} \right\} dx$$

simplify,

$$\int sinh^{13}\, 3x\, dx = \left(\frac{1}{2}\right)^{13} \int \left\{ e^{39x} - \binom{13}{1} e^{33x} + \binom{13}{2} e^{27x} - \binom{13}{3} e^{21x} + \binom{13}{4} e^{15x} - \binom{13}{5} e^{9x} + \binom{13}{6} e^{3x} - \binom{13}{7} e^{-3x} + \binom{13}{8} e^{-9x} - \binom{13}{9} e^{-15x} + \binom{13}{10} e^{-21x} - \binom{13}{11} e^{-27x} + \binom{13}{12} e^{-33x} - e^{-39x} \right\} dx$$

Since $\binom{13}{1} = \binom{13}{12}, \binom{13}{2} = \binom{13}{11}, \binom{13}{3} = \binom{13}{10}, \binom{13}{4} = \binom{13}{9}, \binom{13}{5} = \binom{13}{8}, \binom{13}{6} = \binom{13}{7}$, regroup the terms in the integrand

$$\int sinh^{13}\, 3x\, dx = \left(\frac{1}{2}\right)^{13} \int \left\{ (e^{39x} - e^{-39x}) - \binom{13}{1} [e^{33x} - e^{-33x}] + \binom{13}{2} [e^{27x} - e^{-27x}] - \binom{13}{3} [e^{21x} - e^{-21x}] + \binom{13}{4} [e^{15x} - e^{-15x}] - \binom{13}{5} [e^{9x} - e^{-9x}] + \binom{13}{6} [e^{3x} - e^{-3x}] \right\} dx$$

Take out $\frac{1}{2}$ in $\left(\frac{1}{2}\right)^{13}$, and distribute in the integrand,

$$\int sinh^{13}\, 3x\, dx = \left(\frac{1}{2}\right)^{12} \int \left\{ \frac{(e^{39x} - e^{-39}\ )}{2} - \binom{13}{1} \frac{[e^{33x} - e^{-33x}]}{2} + \binom{13}{2} \frac{[e^{27x} - e^{-27x}]}{2} - \binom{13}{3} \frac{[e^{21x} - e^{-21x}]}{2} + \binom{13}{4} \frac{[e^{15x} - e^{-15x}]}{2} - \binom{13}{5} \frac{[e^{9x} - e^{-9x}]}{2} + \binom{13}{6} \frac{[e^{3x} - e^{-3x}]}{2} \right\} dx$$

Since $sinh\, ax = \frac{e^{ax} - e^{-ax}}{2}$, then the equation above can be written as

$$\int sinh^{13}\, 3x\, dx = \left(\frac{1}{2}\right)^{12} \int \left\{ \sinh 39x - \binom{13}{1} [\sinh 33x] + \binom{13}{2} [\sinh 27x] - \binom{13}{3} [\sinh 21x] + \binom{13}{4} [\sinh 15x] - \binom{13}{5} [\sinh 9x] + \binom{13}{6} [\sinh 3x] \right\} dx$$

Then integrate,

$$\int sinh^{13}\, 3x\, dx = \left(\frac{1}{2}\right)^{12} \left\{ \frac{1}{39} \cosh 39x - \frac{13}{33} [\cosh 33] + \frac{78}{27} [\cosh 27x] - \frac{286}{21} [\cosh 21x] + \frac{715}{15} [\cosh 15x] - \frac{1287}{9} [\cosh 9x] + \frac{1716}{3} [\cosh 3x] \right\} + C$$

Factor out $\frac{1}{3}$, then it gives

$$\int \sinh^{13} 3x\, dx = \left(\frac{1}{3}\right)\left(\frac{1}{2}\right)^{12} \left[\frac{1}{13} \cosh 39x - \frac{13}{11} \cosh 33x + \frac{78}{9} \cosh 27x - \frac{286}{7} \cosh 21x + \frac{715}{5} \cosh 15x - \frac{1287}{3} \cosh 9x + 1716 \cosh 3x \right] + C$$

By the derived general formula:

Solution: Apply formula (3), let $n = 13,\ a = 3,\quad k = 0, 1, 2, 3, 4, 5, 6$

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$$\int \sinh^{13} 3x\, dx = \left(\frac{1}{3}\right)\left(\frac{1}{2}\right)^{13-1}\left[\binom{13}{0}\left(\frac{1}{13-2(0)}\right)\cosh 3(13-2(0))x - \binom{13}{1}\left(\frac{1}{13-2(1)}\right)\cosh 3(13-2(1))x + \binom{13}{2}\left(\frac{1}{13-2(2)}\right)\cosh 3(13-2(2))x - \binom{13}{3}\left(\frac{1}{13-2(3)}\right)\cosh 3(13-2(3))x + \binom{13}{4}\left(\frac{1}{13-2(4)}\right)\cosh 3(13-2(4))x - \binom{13}{5}\left(\frac{1}{13-2(5)}\right)\cosh 3(13-2(5))x + \binom{13}{6}\left(\frac{1}{13-2(6)}\right)\cosh 3(13-2(6))x\right] + C$$

Simplify,

$$\int \sinh^{13} 3x\, dx = \left(\frac{1}{3}\right)\left(\frac{1}{2}\right)^{12}\left[\frac{1}{13}\cosh 39x - \frac{13}{11}\cosh 33x + \frac{78}{9}\cosh 27x - \frac{286}{7}\cosh 21x + \frac{715}{5}\cosh 15x - \frac{1287}{3}\cosh 9x + 1716\cosh 3x\right] + C$$

As shown in the solution above, the used of the derived formula is much simpler and easier.

Example 2: Evaluate $\int \sinh^{26} 4x\, dx$ using the derived formula for even power of hyperbolic sine

Solution: Apply formula (5), let $n = 26,\ a = 4,\ k = 0, 1, 2, 3, \ldots, 12$

$$\int \sinh^{26} 4x\, dx = \left(\frac{1}{4}\right)\left(\frac{1}{2}\right)^{26-1}\left\{\frac{1}{26}\sinh 4x(26) - \binom{26}{1}\left[\frac{1}{(26-2)}\sinh 4x(26-2)\right] + \binom{26}{2}\left[\frac{1}{(26-4)}\sinh 4x(26-4)\right] - \binom{26}{3}\left[\frac{1}{(26-6)}\sinh 4x(26-6)\right] + \binom{26}{4}\left[\frac{1}{(26-8)}\sinh 4x(26-8)\right] - \binom{26}{5}\left[\frac{1}{(26-10)}\sinh 4x(26-10)\right] + \binom{26}{6}\left[\frac{1}{(26-12)}\sinh 4x(26-12)\right] - \binom{26}{7}\left[\frac{1}{(26-14)}\sinh 4x(26-14)\right] + \binom{26}{8}\left[\frac{1}{(26-16)}\sinh 4x(26-16)\right] - \binom{26}{9}\left[\frac{1}{(26-18)}\sinh 4x(26-18)\right] + \binom{26}{10}\left[\frac{1}{(26-20)}\sinh 4x(26-20)\right] - \binom{26}{11}\left[\frac{1}{(26-22)}\sinh 4x(26-22)\right] + \binom{26}{12}\left[\frac{1}{(26-24)}\sinh 4x(26-24)\right]\right\} - \left[\left(\frac{1}{2}\right)\binom{26}{13}x\right] + C$$

simplify,

$$\int \sinh^{26} 4x\, dx = \left(\frac{1}{2}\right)^{27}\left\{\frac{1}{26}\sinh 104x - \frac{26}{24}\sinh 96x + \frac{325}{22}\sinh 88x - \frac{2600}{20}\sinh 80x + \frac{14950}{18}\sinh 72x - \frac{65700}{16}\sinh 64x + \frac{230230}{14}\sinh 56x - \frac{657800}{12}\sinh 48x + \frac{1562275}{10}\sinh 40x - \frac{3124550}{8}\sinh 32x + \frac{5311735}{6}\sinh 24x - \frac{7726160}{4}\sinh 16x + \frac{9657700}{2}\sinh 8x\right\} - \left[\left(\frac{1}{2}\right)10400600x\right] + C$$

Factor out $\frac{1}{2}$ in the sine series, it gives

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$$\int \sinh^{26} 4x\, dx = \left(\frac{1}{2}\right)^{28} \left\{\frac{1}{13}\sinh 104x - \frac{26}{12}\sinh 96x + \frac{325}{11}\sinh 88x - \frac{2600}{10}\sinh 80x + \frac{14950}{9}\sinh 72x - \frac{65700}{8}\sinh 64x + \frac{230230}{7}\sinh 56x - \frac{657800}{6}\sinh 48x + \frac{1562275}{5}\sinh 40x - \frac{3124550}{4}\sinh 32x + \frac{5311735}{3}\sinh 24x - \frac{7726160}{2}\sinh 16x + 9657700\sinh 8x\right\} - [5200300x] + C$$

Example 3: Evaluate $\int \cosh^{15} 2x\, dx$ using the derived formula for odd power of hyperbolic cosine

Solution: Apply formula (8), let $n = 15,\ a = 2,\ \ k = 0, 1, 2, 3, 4, 5, 6, 7$

$$\int \cosh^{15} 2x\, dx = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^{15-1}\left[\binom{15}{0}\left(\frac{1}{15-2(0)}\right)\sinh 2(15-2(0))x + \binom{15}{1}\left(\frac{1}{15-2(1)}\right)\sinh 2(15-2(1))x + \binom{15}{2}\left(\frac{1}{15-2(2)}\right)\sinh 2(15-2(2))x + \binom{15}{3}\left(\frac{1}{15-2(3)}\right)\sinh 2(15-2(3))x + \binom{15}{4}\left(\frac{1}{15-2(4)}\right)\sinh 2(15-2(4))x + \binom{15}{5}\left(\frac{1}{15-2(5)}\right)\sinh 2(15-2(5))x + \binom{15}{6}\left(\frac{1}{15-2(6)}\right)\sinh 2(15-2(6))x + \binom{15}{7}\left(\frac{1}{15-2(7)}\right)\sinh 2(15-2(7))x\right] + C$$

Simplify,

$$\int \cosh^{15} 2x\, dx = \left(\frac{1}{2}\right)^{15}\left[\frac{1}{15}\sinh 30x + \frac{15}{13}\sinh 26x + \frac{105}{11}\sinh 22x + \frac{455}{9}\sinh 18x + \frac{1365}{7}\sinh 14x + \frac{3003}{5}\sinh 10x + \frac{5005}{3}\sinh 6x + 6435\sinh 2x\right] + C$$

Conclusions and Recommendations

Integration formula for the power of hyperbolic sine function of the form $\int sinh^n ax\, dx$, and hyperbolic cosine function of the form $\int cosh^n\, ax\, dx$ where $n \in \mathbb{Z}^+$ and $\forall\, a \in \mathbb{R}$ were derived through the Binomial Theorem. A generalized formula for integrating the odd and even power of the hyperbolic sine function was developed to shorten and simplify the solutions in evaluating such integrals. The derived general formula presented in this paper demonstrated the easiest way to evaluate the integrals of the powers of the hyperbolic sine and the hyperbolic cosine functions. Also, the derived formula can be useful to engineering and mathematics students in solving problems in Differential Equations and other higher mathematics courses. Applications of the general formula were illustrated for a clearer understanding of the derived formulas. It is recommended that the derived formula be used in evaluating integrals that involve large power of hyperbolic sine and hyperbolic cosine functions. Likewise, it is also recommended to conduct further studies to derive algorithms in the integration of the power of hyperbolic tangent, hyperbolic secant, hyperbolic cosecant and hyperbolic cotangent functions.

REFERENCES

Alcantara, E. C. (2015). On the Derivation of Some Reduction Formula Through Tabular Integration by Parts. Asia Pacific Journal of Multidisciplinary Research, 3(1), 80-84. http://surl.li/unioc

Baskonus, H. M., & Bulut, H. (2015). New Hyperbolic Function Solutions for Some Nonlinear Partial Differential Equation Arising in Mathematical Physics. Entropy. 17(6):4255-4270. https://doi.org/10.3390/e17064255

Dar, S. A., & Paris, R. B. (2019). On integrals involving quotients of hyperbolic functions. https://doi.org/10.48550/arXiv.1903.08487

Domondon, C., Pardo, C., & Rin, E. (2022). Analysis of Difficulties of Students in Learning Calculus. Science International. 1-4. http://surl.li/unioh

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P - ISSN 2984-7842
E - ISSN 1908-3181

Kiat, S. E. (2005). Analysis of Students' Difficulties in Solving Integration Problems. Shuqun Secondary School, Singapore. The Mathematics Educator 2005, Vol. 9, No.1, 39-59. http://surl.li/unipv

Stakhov, A., & Rozin, B. (2005). On a new class of hyperbolic functions. Chaos, Solitons & Fractals, Volume 23, Issue 2, Pages 379-389, ISSN 0960-0779. https://doi.org/10.1016/j.chaos.2004.04.022.

Stewart, J., Clegg, D., & Watson, S. (2019). Calculus Early Transcendentals Ninth Edition. Cengage 200 Pier Four Boulevard, Boston, MA 02210, USA.

Stewart, J., Redlin, L., & Watson, S. (2012). Stewart's Algebra and Trigonometry. Cengage Learning Asia Pte Ltd #02-08 New Tech Park Singapore

Thomas, G. (2004). Calculus Early Transcendental. Addison-Wesely, Pearson.

Yerizon, S. F. (2019). Analysis of Difficulty Learning Calculus Subject for Mathematical Education Students. International Journal of Scientific & Technology Research Volume 8, Issue 03, March 2019 Issn 2277-8616 80. http://surl.li/uniql